

Solids of Revolution with Minimum Surface Area

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Abstract We consider the problem of determining the minimum surface area of solids obtained when the graph of a differentiable function is revolved about horizontal lines. We describe two solutions for this problem, one that is cumbersome but instructive, and one that is both elegant and amusing. Maple [2] implementations of the solutions are discussed and several potential difficulties are identified.

1 Introduction

The surface area problem considered in this paper was motivated by a simpler and entertaining problem. Suppose that we wish to determine the minimum volume among the solids obtained by revolving the graph of a continuous function $y = f(x)$ about horizontal lines $y = k$. For a given value of k , the volume of the solid is given by

$$V(k) = \pi \int_a^b (f(x) - k)^2 dx. \quad (1)$$

Expanding this yields the following parabolic equation in k

$$\frac{V(k)}{\pi} = \int_a^b (f(x))^2 dx - 2k \int_a^b f(x) dx + k^2 \int_a^b dx. \quad (2)$$

The vertex of this parabola yields the desired minimum volume corresponding to

$$k = \frac{1}{b-a} \int_a^b f(x) dx. \quad (3)$$

k is thus equal to the average value of $f(x)$ on the interval $[a, b]$.

Let's be adventuresome and consider the question of minimizing the surface area $S(k)$. Let $[f_{\min}$ and $f_{\max}]$ denote the minimum and maximum values of $f(x)$ on the interval $[a, b]$. If k is outside the interval $[f_{\min}, f_{\max}]$, we have

$$\frac{S(k)}{2\pi} = \int_a^b (f(x) - k) \sqrt{1 + (f'(x))^2} dx \quad (4)$$

or its negative. $S(k)$ is thus a piecewise linear function for k not in $[f_{\min}, f_{\max}]$. We're interested in what happens if $f_{\min} < k < f_{\max}$ in which case the surface area is given by

$$\frac{S(k)}{2\pi} = \int_a^b |f(x) - k| \sqrt{1 + (f'(x))^2} dx. \quad (5)$$

The presence of the absolute value complicates things quite a bit as we will see.

Here is a quick summary of the remainder of this paper. We will see that $S(k)$ resembles a parabola on the interval $[f_{\min}, f_{\max}]$ and has a unique minimum (except in the trivial case in which $f(x)$ is a constant function). In Section 2, we will first show that $S'(k) = 0$ has at least one solution. We will then show that $S(k)$ is concave up in order to see that the minimum is unique. In Section 3 we will describe a simple solution procedure for minimizing the surface area. Finally, in Section 4, we will mention some numerical issues.

To give the reader an idea of the shape of the surface area function $S(k)$ for a typical function, Figure 1 depicts $S(k)$ for the function $f(x) = \cos(x)$ on the interval $[0, \frac{\pi}{4}]$. For $k \leq f_{\min} = \frac{\sqrt{2}}{2}$ and for $k \geq f_{\max} = 1$, the surface area (depicted in green) is linear. For $f_{\min} \leq k \leq f_{\max}$, the surface area (depicted in red) resembles a segment of a parabola. (In the simple case that $f(x)$ is a linear function, the surface area is, in fact, parabolic for k in $[a, b]$.) We will exploit the unimodal shape of $S(k)$ in Section 3.

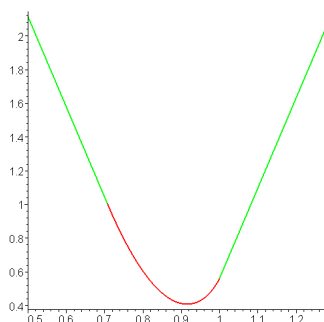


Figure 1: Surface Area $S(k)$ for $f(x) = \cos(x)$

2 Properties of $S(k)$

Before considering the general case, it is instructive to consider the simpler case in which $f(x)$ is monotone. To this end assume that $f(x)$ is decreasing on $[a, b]$. (A similar argument applies to the case in which $f(x)$ is increasing.) For the moment we also assume that $f(x)$ does not have an inflection point in $[a, b]$. In this case

$$\frac{S(k)}{2\pi} = \int_a^c (f(x) - k) \sqrt{1 + (f'(x))^2} dx + \int_c^b (k - f(x)) \sqrt{1 + (f'(x))^2} dx. \quad (6)$$

where $c = f^{-1}(k)$. Differentiating these integrals we obtain, respectively,

$$- \int_a^c \sqrt{1 + (f'(x))^2} dx - (k - f(f^{-1}(k))) \sqrt{1 + (f'(f^{-1}(k)))^2} \quad (7)$$

and

$$\int_c^b \sqrt{1 + (f'(x))^2} dx - (k - f(f^{-1}(k)))\sqrt{1 + (f'(f^{-1}(k)))^2}. \quad (8)$$

The second term in each expression is 0 so we obtain

$$\frac{S'(k)}{2\pi} = - \int_a^c \sqrt{1 + (f'(x))^2} dx + \int_c^b \sqrt{1 + (f'(x))^2} dx. \quad (9)$$

Since $S'(a)$ and $S'(b)$ have opposite signs, there must be at least one solution of $S'(k) = 0$ in (a, b) .

We also have

$$\frac{S''(k)}{2\pi} = 2\sqrt{1 + (f'(c))^2} (f^{-1})'(k) dx \quad (10)$$

or, equivalently,

$$\frac{S''(k)}{2\pi} = -2 \frac{\sqrt{1 + (f'(c))^2}}{f'(c)}. \quad (11)$$

Since $f(x)$ is decreasing, $f'(k) < 0$ and we see that S'' is positive. $S(k)$ is therefore concave up. In the event that $S(k)$ has an inflection point at $(k, f(k))$ so that $f'(c) = 0$, $S''(k)$ approaches $+\infty$. This shows that the solution $S'(k) = 0$ is unique and that it yields an absolute minimum surface area.

Note that this value of k is the one that bisects the arc length of the function from a to b . It is therefore straightforward to implement the solution of $S'(k) = 0$ in a CAS such as Maple. The Maple `fsolve` command can be used in conjunction with the following procedure to solve $S'(k) = 0$. In this procedure `finv` denotes $f^{-1}(x)$.

```
SP0 := proc(k)
  # Calculate the difference of the arc lengths
  # determined by c = finv(k). This is the
  # residual procedure for fsolve.
  local Aval, ALval, ARval, c, c1, c2:
  global f, fp, finv, a, b:
  c := evalf(finv(k)):
  ALval := evalf(Re(int(sqrt(1+(fp(x))^2), x=a..c))):
  ARval := evalf(Re(int(sqrt(1+(fp(x))^2), x=c..b))):
  Aval := ALval - ARval:
  return(Aval):
end proc:
```

For the function $f(x) = \cos(x)$ with $0 \leq x \leq \frac{\pi}{4}$, a minimum surface area of approximately 0.41 is obtained for $k = 0.915$. Figure 2 depicts the corresponding solid. The figure (as well as others given below) was obtained using the Maple `SurfaceOfRevolution` command. For a function $f(x)$ defined on an interval $[a, b]$, this command may be used as

```
SurfaceOfRevolution(f(x), x=a..b, axis=horizontal,
  distancefromaxis=k):
```

for the numerical value of the surface area and

```
SurfaceOfRevolution(f(x), x=a..b, axis=horizontal,
distancefromaxis=k, output=plot):
```

for graphical output.

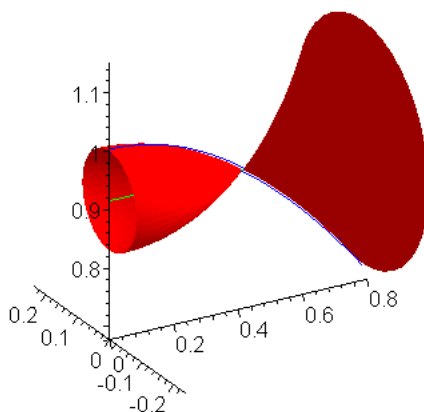


Figure 2: Minimum Surface of Revolution for $f(x) = \cos(x)$

To see what needs to be done in the event $f(x)$ is not monotone, let's consider the next easiest case in which $f'(x) = 0$ has exactly one solution c in (a, b) . Let c_1 and c_2 be the values for which $f(c_1) = k$ and $f(c_2) = k$ with $c_1 < c$ and $c_2 > c$. We assume that $f(x)$ starts above the line $y = k$, that (c_i, k) is not an inflection point, and that $f(x) - k$ changes sign at each of c_1 and c_2 . Note that if the latter condition is not satisfied, the integrand in $S(k)$ does not change sign at $x = c_i$; so c_i can be ignored in the integrals defining $S(k)$. We denote by $f_1^{-1}(x)$ and $f_2^{-1}(x)$ the local inverses of $f(x)$ near $x = c_1$ and $x = c_2$, respectively.

With these assumptions, the surface area is given by

$$\begin{aligned} \frac{S(k)}{2\pi} = & \int_a^{c_1=f_1^{-1}(k)} (f(x) - k) \sqrt{1 + (f'(x))^2} dx - \\ & \int_{c_1=f_1^{-1}(k)}^{c_2=f_2^{-1}(k)} (f(x) - k) \sqrt{1 + (f'(x))^2} dx + \\ & \int_{c_2=f_2^{-1}(k)}^b (f(x) - k) \sqrt{1 + (f'(x))^2} dx. \end{aligned} \tag{12}$$

As in the monotone case, we can calculate $S'(k)$ and $S''(k)$. Using the fact that $f_1^{-1}(x)$ is decreasing at $x = c_1$ and $f_2^{-1}(x)$ is increasing at $x = c_2$, we see that $S(k)$ is concave up. As before, a similar

argument is applicable if $f(x)$ starts below the line $y = k$ or $f(a) = k$; and $S''(k)$ approaches $+\infty$ if (c_i, k) is an inflection point.

It is more difficult to directly solve $S'(k) = 0$ in this case. To avoid the recursive use of `fsolve`, we can use a Maple adaptation `Zeromw.mws` of the Zero root finder from [3]. (Zero uses a combination of bisection and the secant method to solve nonlinear equations.) For the function $f(x) = x^3 - 3x^2 + 2x + 3$ on the interval $(1, 2)$, the following is a procedure which may be used in conjunction with `Zeromw.mws` to solve $S'(k) = 0$. This procedure uses `fsolve` to calculate each of c_1 and c_2 for a given value of k and defines $S'(k)$. For this function, a minimum surface area of approximately 0.86 is obtained for $k = 2.77$. Figure 3 depicts the corresponding solid.

```
SP0Z := proc(k)
  # Calculate the difference of the arc lengths determined
  # by f(c) = k. This is the residual procedure for Zeromw.
  global a, b;
  local Aval1, Aval2, Aval3, Aval, c1, c2;
  fsolve(f(c1)=k, {c1}, a..(3+sqrt(3))/3):
  assign(%):
  evalf(%):
  c1 := Re(c1);
  fsolve(f(c2)=k, {c2}, (3+sqrt(3))/3..b):
  assign(%):
  evalf(%):
  c2 := Re(c2);
  Aval1 := evalf(Re(int(sqrt(1+(fp(x))^2), x=a..c1))):
  Aval2 := evalf(Re(int(sqrt(1+(fp(x))^2), x=c1..c2))):
  Aval3 := evalf(Re(int(sqrt(1+(fp(x))^2), x=c2..b))):
  Aval := 'Aval':
  Aval := Aval1 - Aval2 + Aval3:
  return(Aval):
```

Directly solving $S'(k) = 0$ isn't really feasible in general. Consider, for example, the function $f(x) = x^3 \sin(1/x) + 1$ with $0.05 \leq x \leq 0.2$ as depicted in Figure 4. The number of solutions of $f(x) = k$ varies from 1 to 5 as k is varied. Although it would be possible to determine the c_i , doing so would be cumbersome at best. In the next section we will describe a much simpler solution that is applicable to this function as well as more general ones.

A few words are in order regarding the general case. It is worth emphasizing that our proof depends on the way in which the c_i are chosen: c_i for which $f'(c_i) = 0$ and $f''(c_i) \neq 0$ are ignored. For example, consider the function $f(x) = -x^3 + 5x^2 - 7x + 1$ with $a = 0 \leq x \leq b = 3.5$ as depicted in Figure 5. For $k = -2$ there are two solutions of $f(c) = k$, $c = 1$ and $c = 3$. $f'(1) = 0$ and this value yields a relative minimum. We can therefore ignore this value since $k - f(x)$ does not change sign at $c = 1$. Rather than use three integrals (from a to 1, from 1 to 3, and from 3 to b) to define $S(k)$, we can use two integrals (from a to $c_1 = 3$ and from c_1 to b) to do so.

The above arguments can be extended to the general case. We assume that the graph of $y = f(x)$ intersects the line $y = k$ finitely many times and that $f'(x) = 0$ has finitely many solutions for $f_{\min} < k < f_{\max}$. We assume the points of intersection are (c_i, k) , $i = 1, \dots, n - 1$ with $c_i < c_{i+1}$,

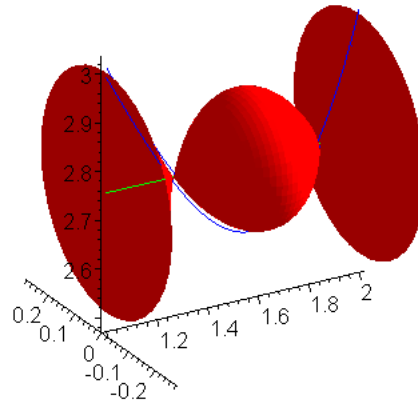


Figure 3: Minimum Surface of Revolution for $f(x) = x^3 - 3x^2 + 2x + 3$

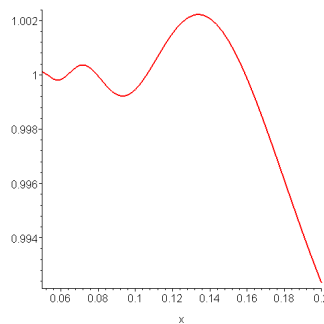


Figure 4: $f(x) = x^3 \sin(1/x) + 1$

and that $f(x) - k$ changes sign at each $x = c_i$ since, as above, the two integrals in $S(k)$ having such a c_i as an endpoint do not contribute to the value of $S''(x)$. We need to determine the contribution to $S''(x)$ at $x = c_i$ and show that it is positive. We will first assume that $f'(c_i) \neq 0$. We will then consider the case in which $f'(c_i) = 0$ so that $(c_i, f(c_i))$ is an inflection point of $f(x)$. We have

$$\frac{S(k)}{2\pi} = \sum_{i=0}^{n-1} (-1)^i \int_{c_i}^{c_{i+1}} (f(x) - k) \sqrt{1 + (f'(x))^2} dx \quad (13)$$

where for convenience we define $c_0 = a$ and $c_n = b$. Differentiating twice we see that

$$\frac{S''(k)}{2\pi} = \sum_{i=1}^{n-1} 2 \sqrt{1 + (f'(c_i))^2} (f_i^{-1})'(k) \quad (14)$$

or its negative (depending on whether $f(x)$ starts above or below the line $y = k$). In either case,

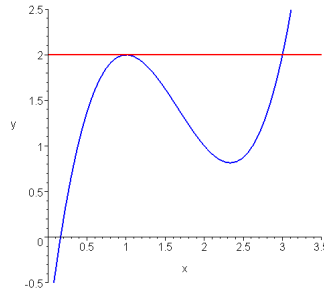


Figure 5: $f(x) = -x^3 + 5x^2 - 7x + 1$

$S''(k) > 0$. To see this, assume $f(x)$ is decreasing at $x = c_i$. The two integrals of interest are then

$$\int_{c_{i-1}}^{c_i=f_i^{-1}(k)} (f(x) - k) \sqrt{1 + (f'(x))^2} dx - \int_{c_i=f_i^{-1}(k)}^{c_{i+1}} (f(x) - k) \sqrt{1 + (f'(x))^2} dx. \quad (15)$$

where f_i^{-1} is the local inverse of $f(x)$ near c_i . The contribution to $S''(x)$ is thus

$$-2 \sqrt{1 + (f'(c_i))^2} (f_i^{-1})'(k) = -2 \frac{\sqrt{1 + (f'(c_i))^2}}{f'(c_i)}. \quad (16)$$

This contribution is positive since $f'(c_i) < 0$. In the event $f(x)$ is increasing at $x = c_i$, the corresponding contribution is

$$2 \frac{\sqrt{1 + (f'(c_i))^2}}{f'(c_i)} \quad (17)$$

which is also positive since $f'(c_i) > 0$. If $(c_i, f(c_i))$ is an inflection point of $f(x)$, $S''(x) \rightarrow +\infty$ as $x \rightarrow c_i$ since either of the above contributions becomes

$$\lim_{x \rightarrow c_i} \sqrt{1 + \frac{1}{(f'(x))^2}}. \quad (18)$$

As with the above simpler cases, we see that $S(k)$ is concave up as claimed.

3 Golden Search Minimization

We want to exploit the fact that there is a unique solution of $S' = 0$ due to the fact that $S(k)$ is concave up. For such a unimodal function, a Golden Search minimization [1], also referred to as a Golden Section or Fibonacci minimization, is applicable. Following is a procedure to evaluate $S(k)$ for use with the Golden Search minimization. In this procedure fp is the derivative $f'(x)$. The procedure is perfectly general and is applicable to any differentiable function. It thus represents an elegant solution for minimizing the surface area.

```

SM := proc(k)
  # Calculate the physical surface area and multiply by -1.
  # This is the function procedure for the Golden Search
  # minimization.
  local Sval:
  global f, fp, a, b:
  Sval :=
  -evalf(2*Pi*Re(int(abs(k-f(x))*sqrt(1+(fp(x))^2), x=a..b))):
  return(Sval):
end proc:

```

For the function $f(x) = x^3 \sin(1/x) + 1$ defined above, a minimum surface area of approximately 0.0016 is obtained for $k = 0.99995$. Figure 6 depicts the corresponding solid. To illustrate the applicability of the Golden Search method to functions for which solving for all values of $f(c) = k$ is not feasible, we can move the left endpoint closer to 0 in which case there may be many solutions. The Golden Search minimization easily finds the minimum surface area for this function. We mention that the Maple optimization command `NLPSolve` can be used to verify the accuracy of the Golden Search minimizations given in this paper.

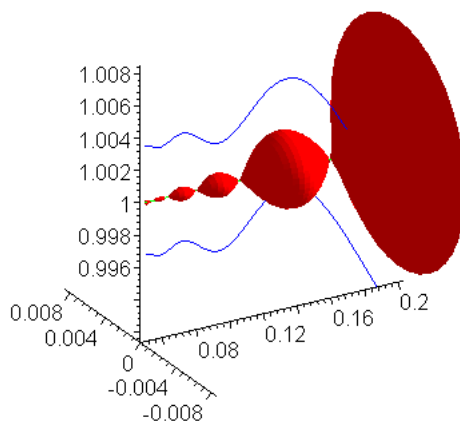


Figure 6: Minimum Surface of Revolution for $f(x) = x^3 \sin(1/x) + 1$

4 Final Comments

Care must be exercised when a CAS such as Maple is used to evaluate the necessary integrals arising in the minimization of S . For example, complex arithmetic is used in the evaluation of the necessary

integrals for $S(k)$ for some functions. Use of the real part command $\text{Re}()$ can be used to minimize the problems associated with this when the imaginary part of the result is essentially 0. For some functions, however, if the integrands are not handled with care, the integrations involving elliptic integrals may well yield incorrect results (*e.g.*, quantities that are clearly negative being calculated as complex numbers with large positive real parts and nonzero imaginary parts).

It is easy to verify these remarks by considering the simple function $f(x) = x^3$ for $-1 \leq x \leq 1$, paying particular attention to the values and graphs of $S(k)$, $S'(k)$, and $S''(k)$ for negative values of k . Readers might be interested in investigating the minimum surface area problem for this function. Figure 7 shows correct plots of $f(x)$, $S(k)$, $S'(k)$, and $S''(k)$ for this function. For this function, a minimum surface area of approximately 7.1 is obtained for $k = 0$.

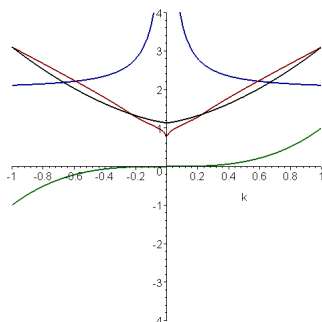


Figure 7: $S(k)$, $S'(k)$, $S''(k)$ for $f(x) = x^3$

Readers interested in the numerical approximation of definite integrals might wish also to investigate the use of purely numerical quadrature algorithms for this problem, for example, the Adapt algorithm from [3].

It is instructive to study the manner in which the solids of revolution change as a function of the right endpoint b . Figures 8 and 9 depict the behavior for the functions $f(x) = e^{-x} \cos(2\pi x)$ and $f(x) = \sqrt{4 - x^2}$.

Maple worksheets are available for generating the various graphs and numerical results given in this paper, for implementing the two solutions discussed, for illustrating potential difficulties requiring careful treatment of the integrands for the required integrals, and for illustrating the use of Maple adaptations of Zero and Adapt for this problem. Refer to the list of supplemental electronic documents at the end of this paper.

We point out that extending the results of this paper to revolving graphs of differentiable functions and (some) parametric equations about oblique lines leads to a new set of surprising and entertaining issues. The results will be discussed in a forthcoming paper [4].

Figure 8. $S(k_{\min}, b)$ for $f(x) = e^{-x} \cos(2\pi x)$

Figure 9. $S(k_{\min}, b)$ for $f(x) = \sqrt{4 - x^2}$

5 Supplemental Electronic Materials

- Thompson, S., Maple worksheet SAK.mws for implementing the Golden Section minimization and the direct solution of $S'(k) = 0$.
- Thompson, S., Maple worksheet SAKA.mws for implementing the Golden Section minimization in conjunction with Adaptmw.mws and the Maple int command for integration.
- Thompson, S., Maple worksheet ShortAndSweet.mws for implementing the Golden Section minimization and integration procedures without the various extraneous plots in SAK.mws and SAKA.mws.
- Thompson, S., Maple worksheet fxcubed.mws for exploring potential problems with the integrations.

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References

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